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Calculation of Stress Intensity Factors for Cracks in Structural and Mechanical Components Subjected to Complex Stress Fields


Abstract: One of the problems in using fracture mechanics is the determination of stress intensity factors of cracked structural and mechanical components. The cracks are often subjected to complex stress fields induced by external loads and residual stresses resulting from the surface treatment. Both stress fields are characterized by non-uniform distributions and handbook stress intensity factor solutions are in such cases seldom available. The method presented below is based on the generalized weight function technique enabling the stress intensity factors to be calculated for any Mode I loading applied to a planar semi-elliptical surface crack. The stress intensity factor can be determined at any point on the crack tip contour by using the general weight function. The calculation is carried out by integrating the product of the stress field and the weight function over the crack area.

Several examples of point-load weight functions and resulting stress intensity factors are presented in the paper. The method is particularly suitable for modeling fatigue crack growth in the presence of complex stress fields.

Keywords: stress intensity factor, weight function, nonlinear stress field

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Nomenclature

\(a\)  
Depth of an edge crack or the shorter semi-axis of an elliptical crack

\(c\)  
The longer semi-axis of an elliptical crack

\(A\)  
Point on the crack contour where the stress intensity factor is to be calculated

\(FE\)  
Finite element analysis

\(G_c\)  
Crack contour

\(G_b\)  
External boundary contour

\(K_I\)  
Model I stress intensity factor (general)

\(K_{IA}\)  
Model I stress intensity factor at the point \(A\) on the crack front

\(M\)  
Geometry correction factor, \(M = \frac{m_A(x,y)}{\sqrt{\pi a}}\)

\(M_i\)  
Coefficients of the 1-D line load weight functions (\(i = 1,2,3\))

\(m(x,a)\)  
Weight function (general)

\(m(x,y)\)  
Weight function for point \(A\) on the crack contour

\(\Omega\)  
Crack area

\(F\)  
Point load (force) applied to the crack surface at point \(P(x,y)\)

\(P(x,y)\)  
Point on the crack surface where the load \(F\) is applied

\(Q\)  
Shape factor, \(Q = 1 + 1.464 \cdot \left(\frac{a}{c}\right)^{1.65}\)

SIF  
Stress intensity factor

\(s\)  
Shortest distance between the point load and the crack contour

\(t\)  
Thickness

\(\Gamma_c\)  
Inverted crack contour

\(\Gamma_b\)  
Inverted free boundary contour

\(\rho\)  
Distance between the point load and the point on the crack front where the stress intensity factor is to be calculated

\(r_A\)  
The radius of the inside circle tangent to the ellipse at the point \(A\) where the stress intensity factor is to be calculated

\(R\)  
The radius of the biggest inside circle tangent to the ellipse

\(\sigma_m\)  
Membrane (tensile) stress

\(\sigma_b\)  
Bending stress

\(\sigma(x)\)  
One-dimensional stress distribution

\(\sigma(x,y)\)  
Two-dimensional stress distribution

\(Y\)  
Geometry correction factor, \(Y = \frac{K}{\sigma_0 \cdot \sqrt{\pi a}}\)
Introduction

Most of the existing methods of calculating stress intensity factors require separate analysis of each load and geometry configuration. Fortunately, the weight function method developed by Bueckner [1] and Rice [2] simplifies considerably the determination of stress intensity factors. The important feature of the weight function is that it depends only on the geometry of the cracked body. If the weight function is known for a given cracked body, the stress intensity factor due to any Model I load system applied to the body can be determined by using the same weight function. If the weight function is known, there is no need to derive ready-made stress intensity factor expressions for each load system and associated internal stress distribution induced in the cracked body. The stress intensity factor for a one-dimensional crack (Fig. 1) can be obtained by multiplying the weight function, \( m(x,a) \), and the internal stress distribution, \( \sigma(x) \), in the prospective crack plane, and integrating the product over the crack length \( a \)

\[
K = \int_0^a \sigma(x) \cdot m(x,a) \, dx \quad (1)
\]

A variety of one-dimensional (line load) weight functions can be found in Refs [3-5]. However, their mathematical forms vary from case to case making their application inconvenient. Therefore Glinka and Shen [6] have proposed a general weight function expression (2), which can be used for a wide variety of one-dimensional Mode I cracks

\[
m(x,a) = \frac{2F}{\sqrt{2\pi(a-x)}} \left[1 + M_1 \left(1 - \frac{x}{a}\right)^{1/2} + M_2 \left(1 - \frac{x}{a}\right) + M_3 \left(1 - \frac{x}{a}\right)^{3/2}\right] \quad (2)
\]

Figure 1. The line load (1-D) weight function notation
The system of coordinates and notation for an edge crack as an example are given in Fig. 1. In order to determine the weight function, \( m(x,a) \), for a particular cracked body, it is sufficient to determine the three parameters \( M_1, M_2 \) and \( M_3 \).

Because the mathematical form of the weight function (2) is the same for all cracks, the same method can be used for the determination of parameters \( M_i \) and calculation of stress intensity factors based on Eq. 1. The method of finding \( M_i \) parameters was discussed in reference [7]. A variety of line load weight functions have been derived and published [6-9] already.

**Point Load Weight Functions for Planar Cracks Subjected to Two-Dimensional Stress Fields**

In spite of the high efficiency and usefulness of one-dimensional line load weight functions, they cannot be used if the stress field is two-dimensional in nature, i.e., when the stress field, \( \sigma(x,y) \), in the crack plane depends on both \( x \) and \( y \) coordinates. Therefore, in order to calculate stress intensity factors for planar cracks subjected to a two-dimensional (2-D) stress field, a weight function for a point load (Fig. 2) is needed. There are some solutions for planar cracks available in the literature [10-11]. For example, stress intensity factors for a semi-elliptical surface crack in a flat plate [10] can be calculated using following equation:

\[
K_i = \left( \sigma_m + H \sigma_h \right) \sqrt{\frac{\pi a}{Q}} f \left( \frac{a}{t}, \frac{a}{c}, \frac{c}{W}, \phi \right)
\]

However, the existing solutions for planar cracks can be used only for specific load conditions such as tension and bending. For arbitrary crack configurations and complex two-dimensional stress fields, a 2-D point load weight function is needed.

The 2-D point-load weight function, \( m(x,y) \), represents the stress intensity factor at point A (Fig. 2) on the crack front, induced by a pair of unit splitting forces, \( F \), applied to the crack surface at point \( P(x,y) \). If such a weight function is known, it is possible to calculate the stress intensity factor at any point on the crack front induced by any Mode I stress fields applied to the crack surface. In order to determine the stress intensity factor induced by a 2-D stress field, \( \sigma(x,y) \), at a point A on the crack front, the product of the stress function, \( \sigma(x,y) \), and the weight function, \( m(x,y) \), needs to be integrated over the entire crack surface area, \( \Sigma \)

\[
K_A = \iint_{\Omega} \sigma(x,y) m(x,y) dxdy
\]
Rice has shown [12] that the 2-D point load weight function for an arbitrary planar crack in an infinite body can be generally written as

$$K_A = m_A(x, y) = \frac{\sqrt{2s}}{\pi^{3/2}} \cdot w(x, y)$$  \hspace{1cm} (5)

Oore and Burns [13] proposed an approximate 2-D point-load weight function (6), from which the function \( w(x, y) \) can be derived for a number of crack configurations.

$$K_A = m_A(x, y) = \frac{\sqrt{2}}{\pi \rho^2} \cdot \int_{\Gamma_c} \frac{dG_c}{\rho^2}$$  \hspace{1cm} (6)

Figure 2 – The point load (2-D) weight function notation

The notation for the weight function (5) is given in Fig. 2. Oore and Burns [13] have shown that after deriving closed form expressions for the line integral in expression (5), several exact weight functions could be derived for straight and circular cracks in infinite bodies. It has also been found that the line integral represents the arc length, \( \Gamma_c \), of the crack contour inverted (Fig. 2) with respect to the point, \( P(x, y) \). As a consequence the weight function (6), can be written in a short form as:

$$K_A = m_A(x, y) = \frac{\sqrt{2}}{\pi \rho^2 \sqrt{\Gamma_c}}$$  \hspace{1cm} (7)

The inverted contour, \( \Gamma_c \), can be also looked at (Fig. 2) as the locus of inverted radii \( 1/\rho_i \). Subsequently, it can also be proved that inverted contours form circles in the case of straight and circular crack contours. In other words, the inverted contour is a circle in the case of cracks with a constant curvature. Therefore, based on Eq.7, it was possible
[14] to derive closed form weight functions for a variety of straight and circular crack configurations.

**Embedded Cracks in Finite Bodies – The External Boundary Effect**

The point load weight function (7) can be used only for cracks in infinite bodies. However, in the case of finite bodies both the crack contour and the free boundary contour have to be considered. The existing closed form point load weight function [3] revealed that the following form of the point load weight function (8) might appropriately account for the free boundary effect

\[
K_A = m_A(x, y) = \frac{\sqrt{2}}{\pi \rho^2} \times \frac{\sqrt{\Gamma_c + \Gamma_b}}{\Gamma_c}
\]  

(8)

The parameter, $\Gamma_b$, is the length of the inverted contour of the free boundary with respect to point A on the crack front (Fig. 3) where the stress intensity factor is to be calculated.

Equation (8) was subsequently used to derive a few specific weight functions for crack configurations available in the literature. The weight function for an infinite straight edge crack approaching a straight free boundary (Fig. 3) is discussed below as an example. The weight function (9) for configuration shown in Fig. 3 was derived directly from Eq. 8

\[
m_A(x, y) = \frac{\sqrt{2}s}{\pi \rho^2} \sqrt{1 + \dfrac{s}{d}}
\]  

(9)

The point load weight function (9) can be further integrated along the line $x = 0$ resulting in the 1-D line load weight function (10) for an edge crack approaching a free straight boundary.

\[
m_A(s) = \sqrt{2} \int_{x=0}^{\infty} \frac{s}{\pi \rho^2} \sqrt{1 + \dfrac{s}{d}} dy = \frac{2}{\sqrt{\pi s}} \sqrt{1 + \dfrac{s}{d}}
\]  

(10)

However, the weight function (10) is valid only for configurations where the bending of the uncracked section is negligible.

The weight function for two edge crack under symmetric loading and separated by a finite thickness ligament (Fig. 4) was analyzed next and resulted in expression (11)

\[
m_A(s, d) = \left(\frac{\sqrt{2}}{\pi \rho_1^2} + \frac{\sqrt{2}}{\pi \rho_2^2}\right) \times \sqrt{\frac{\Gamma_c + \Gamma_b}{\Gamma_c}} = \frac{2s}{\pi^{3/2}} \sqrt{\frac{d + s}{d}} \left(\frac{1}{\rho_1^2} + \frac{1}{\rho_2^2}\right)
\]  

(11)
Integration of the point load weight function (11) for uniformly distributed line load $P$ along the two lines of $x = \pm \left(\frac{d}{2} + s\right)$ resulted in the line weight function (12).

$$m_A(s) = \frac{\sqrt{2}}{\sqrt{\pi s}} \times \frac{d + 2s}{\sqrt{d(d + s)}}$$  \hspace{1cm} (12)

The weight function (12) is the same as the exact analytical line weight function derived by Tada [3].

**Figure 3 - An infinite crack approaching a free boundary**

**Figure 4 - Two infinite edge cracks loaded symmetrically and separated by a finite thickness ligament $d$**

*Edge Crack in Infinite Body – Crack Mouth Correction*
When the crack has a finite crack depth (Fig. 5), the effect of the crack mouth boundary has to be taken into account. The general weight function (8) can be used to derive the point load weight function for such a crack configuration as well.

The point load weight function for the configuration shown in Fig. 5 can be written in the form of Eq. 13. The last term in Eq. 13 is the correction for the crack mouth effect. The crack mouth correction can be achieved by applying virtual force symmetric with respect to the crack mouth. The distance between the point where the virtual symmetric load is applied and the point where the stress intensity factor is to be calculated is $\rho_3$

$$m_A(x, y) = \left(\frac{\sqrt{2}}{\pi \rho_1^2} + \frac{\sqrt{2}}{\pi \rho_2^2}\right) \cdot \frac{\sqrt{\Gamma_b + \Gamma_c}}{\Gamma_c} + \frac{1}{\pi \rho_3^2 \sqrt{\Gamma_c}}$$  \hspace{1cm} (13)

Integration of Eq. 13 along line $x = \pm (d + s)$ resulted in the derivation of the line load weight function (14) for a double edge crack subjected to symmetric loading (Fig. 5)

$$m_A(x) = \frac{2(d + s)}{\sqrt{\pi ds}(s + 2d)} + \frac{\sqrt{s}}{\sqrt{\pi} (2a - s)}$$  \hspace{1cm} (14)

Figure 5 - Double edge crack in a finite width plate

When the thickness of the ligament tends to infinity, the weight function (14) takes the form of the weight function for an edge crack in a semi-infinite plate given in Ref [3]. The comparison of the weight function (14) based stress intensity factor correction
factor, $M = m_A(x, y)/\sqrt{\pi a}$, with Tada’s solution [3] is shown in Fig. 6. The good agreement indicates that crack mouth correction in the weight function (14) can approximately account for the crack mouth boundary effect.

\[
\Gamma(x) = \left[ \frac{\sqrt{2}}{\pi \rho_1^2} + \frac{\sqrt{2}}{\pi \rho_2^2} \right] \cdot \frac{\sqrt{\Gamma_b + \Gamma_c \cdot (a/r_A)^0.5}}{\Gamma_c \cdot (a/r_A)^0.5} + \frac{1}{\pi \rho_3^2 \cdot \sqrt{\Gamma_c \cdot (a/r_A)^0.5}}
\]

Parameter $r_A$ is the radius of an internal circle (Fig. 7) tangent to the ellipse at point A.

**Planar Cracks with Variable Crack Front Curvature - The Curvature Effect**

It was also found that the accuracy of the weight function (8) and subsequent accuracy of stress intensity factors for elliptical cracks, having varying curvature of its contour was decreasing as the ellipses became more slender, i.e., when they departed significantly from the circular constant curvature contour. It was concluded that the inverted crack contour, $\Gamma_c$, in Eq. 7 is only an average measure of the crack geometry effect. The weight function and the stress intensity factor depend also on the immediate curvature surrounding the point where the stress intensity factor is to be calculated (Fig. 7) and the proximity of the other parts of the crack contour. The correction for the local curvature effect proposed below is empirical in nature and was deducted from the stress intensity data for a wide variety of stress intensity factors for semi-elliptical surface cracks and general properties of the weight function

\[
m_A(x, y) = \left( \frac{\sqrt{2}}{\pi \rho_1^2} + \frac{\sqrt{2}}{\pi \rho_2^2} \right) \cdot \frac{\sqrt{\Gamma_b + \Gamma_c \cdot (a/r_A)^0.5}}{\Gamma_c \cdot (a/r_A)^0.5} + \frac{1}{\pi \rho_3^2 \cdot \sqrt{\Gamma_c \cdot (a/r_A)^0.5}}
\]

\[
WF14 \quad \text{Ref 3}
\]

**Figure 6 - Comparison of the geometric correction factor, $M$, calculated using the weight function (14) with Tada’s solution [3]**

**Stress Intensity Factors for a Pair of Semi-elliptical Surface Cracks in a Finite Thickness Plate**
Using the point load weight function (15), one can determine the stress intensity factor $K$ for a pair of symmetric semi-elliptical cracks in a finite thickness plate (Fig. 7). The stress intensity factors at the deepest point C (Fig. 8) were determined for a uniform tensile stress field, $\sigma(x,y) = \sigma_0 = 1$ using numerical integration of the weight function (15). The comparison of calculated stress intensity factors in terms of the geometric correction factor, $Y = K / (\sigma_0 \cdot \sqrt{ma})$, with Isida et al [15] data for $a/t = 0.5$ is presented as an example in Fig. 8. For relative crack depths within the range of $0.2 < a/t < 0.8$, and the aspect ratio of $0.2 \leq a/c \leq 1$ the maximum difference between analogous two sets of data was 7.9%.

Figure 7 – A finite thickness plate with a pair of symmetric semi-elliptical surface Cracks

Figure 8 – Comparison of the weight function based SIFs with Isida, et. al data [15] for various aspect ratios $a/c$ and the relative depth of $a/t = 0.5$

Stress Intensity Factors for a Single Semi-Elliptical Surface Crack in a Finite Thickness Plate
The notation for a semi-elliptical surface crack in a finite thickness plate is shown in Fig. 9. The weight function (14) was used for the determination of the stress intensity factor for this crack configuration.

![Figure 9 - Semi-elliptical surface crack in a finite thickness plate](image1)

Two virtual symmetric loads were used to account for the free boundary and the crack mouth effect. The weight function (14) gave good stress intensity factor estimations for semi-circular surface crack \( a/c = 1 \) with relative crack depths \( 0 < a/t < 0.8 \). The error was less than a few percent for the two non-uniform stress fields used for the analysis, i.e., \( \sigma(x, y) = \sigma_0 x/c \) (Fig. 10) and \( \sigma(x, y) = \sigma_0 y/ac \) (Fig. 11).

![Figure 10 – 2-D stress field, \( \sigma(x, y) = \sigma_0 x/c \) applied to the crack surface](image2)
When the crack front approaches the free surface, the weight function based stress intensity factor deviates from the FE data of Ref. [11]. One of the reasons is that numerical integration technique was used to deal with singularities for which our integration was not sufficiently accurate. The integration was accurate enough over the region defined by the parametric angle of $5^0 \leq \theta \leq 175^0$. The comparison of the geometric correction factor $Y$ obtained from the weight function (15) with the FE data of Ref [11] are shown in Figures 12-14. For a single surface crack in a finite thickness plate, the weight function (15) yields good results for relatively deep cracks ($0 < a/t < 0.8$) with aspect ratio of $a/c > 0.5$. 

Figure 11 – 2-D stress field, $\sigma(x, y) = \sigma_0 \times y/ac$ applied to the crack surface

Figure 12 - Comparison of the weight function based geometric SIF correction factor $Y$ with Nilsson’s [11] FE data, $[\sigma(x, y) = \sigma_0 \times y/c, a/c = 1, a/t=0.8]$
Unfortunately the weight function (15) requires an additional term accounting for the effect of bending occurring in the case of long cracks with aspect ratio of $a/c < 0.3$ and $a/t > 0.5$. Therefore, further studies are being carried out in order to include the bending effect in edge cracks ($a/c \to 0$) and semi-elliptical cracks with the aspect ratio of $a/c < 0.3$. The weight function (15) yields good results for embedded elliptical and symmetric edge and semi-elliptical cracks.

![Figure 13](image1)

**Figure 13** - Comparison of the weight function based SIFs with Nilsson’s [11] FE data, 
$[\sigma(x,y) = \sigma_0 \cdot xy/ac, a/c = 1, a/t = 0.8]$

![Figure 14](image2)

**Figure 14** - Comparison of the weight function based SIFs with Nilsson’s [11] FE data, 
$[\sigma(x,y) = \sigma_0 \cdot y/a, a/c = 0.5, a/t = 0.4]$
Conclusions:

The point load weight function (15) gives good estimation of stress intensity factors of double symmetric edge crack and a semi-elliptical surface crack at the deepest point in a finite thickness plate. The stress intensity factors obtained from weight function (15) were compared with the results obtained from finite element analyses [11]. The difference between the stress intensity factors obtained from the finite element analysis and the stress intensity factors calculated from weight function (15) is generally within a few percent for double semi-elliptical surface cracks with the aspect ratio of \( a/c > 0.25 \) and relative crack depth of \( 0 < a/t < 0.8 \).

The point load weight function (15) can be used to calculate stress intensity factors at any point along the crack front for semi-elliptical surface cracks subjected to 2-D stress distribution. The difference between stress intensity factors obtained from the weight function (15), the Handbook solutions [15], and finite element results [11] are a few percent for semi-elliptical surface cracks depth of \( 0 < a/t < 0.8 \), and aspect ratio of \( a/c \geq 0.5 \).

The limitation of the point load weight function (15), is that the relative crack depth of the crack. When relative crack depth of the crack \( a/t > 0.4 \), the stress intensity factors obtained from the weight function (15) deviated from the finite element results. Additional research is needed to account for the local bending effect.

References


