Dealing with Multiaxial Loading

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25 October 2006
Borås, Sweden
Outline

• Motivation
  – Tire Loading
  – Observation of crack formation

• Physical connection of SED to FM

• CED as available energy density

• Comparison of CED to LEFM

• Scalar parm under rotation

• CED is available energy density

• It is an approximation to FM
Strain History in a Tire

Green LaGrange Strain, $E_{ij}$

- $E_{11}$
- $E_{22}$
- $E_{33}$
- $E_{12}$
- $E_{23}$
- $E_{31}$
Axial / Shear Fatigue Experiments

NR + 60 phr N650

Test Specimen

Loading Paths Investigated

Cracking Occurs on Specific Material Plane(s)!

Path B
Specimen 101
\( N_f = 134739 \)

Path D
Specimen 73
\( N_f = 58722 \)

Path C
Specimen 122
\( N_f = 256648 \)

Path E
Specimen 163
\( N_f = 247440 \)

Path J
Specimen 192
\( N_f = 2848 \)

Path A
Specimen 61
\( N_f = 51595 \)

Path K
Specimen 190
\( N_f = 6615 \)
Fatigue Life Analysis

- Load History Applied to Component
- Load History Experienced at critical location, on critical plane
- Fatigue Life
Life Calculation Scheme

\[ F(t) \]

FEA

\[ \varepsilon_{ij}(t), \sigma_{ij}(t) \]

Equivalence Parameter
(Loading Conditions Experienced on a material plane)

\[ \vec{r}(\theta, \phi) \]

Energy Release Rate
(Loading Conditions Experienced on at Crack Tip)

\[ \frac{dc}{dN} = f(T, R) \]

\[ N_f(\theta, \phi) = \int_{c_0}^{c_f} \frac{1}{f(T, R)} \, dc \]

\[ \theta_{\min}, \phi_{\min}, N_{f,\min} \]
Can we relate a nucleation-oriented equivalence parameter to a fracture-mechanics-oriented parameter?

\[
\frac{\text{Energy}}{\text{Area}} = \frac{\text{Energy}}{\text{Volume}} \times \text{Length}
\]
Connection of SED to FM

\[ dU = -W(4\pi r^2 dr) \]

\[ T = -\frac{dU}{dA} = 2Wr \]

\[ dA = 2\pi rdr \]
Limitation of SED

Highly Strained Region

Unstrained Region

Nominal Strain Region

Crack

Highly Strained Region
Cracking Energy Density

\[ dW_c = \sigma \cdot d\varepsilon \]

Proposed Estimator of ERR

\[ T = -\frac{dU}{dA} \approx C W_c \ a \]
Plane vs. Scalar Parameters

Plane Specific Continuum Parameter

\[ T_{Wc} = (2kW_c)\alpha \]

- CED approximately quantifies the portion of SED available on a given material plane.

Scalar Continuum Parameters

\[ T_\varepsilon = (C\varepsilon^2)\alpha \]
\[ T_W = (2kW)\alpha \]

- Max Prin Strain (or stress) refers to a direction associated with the loading state, not a plane in the material.
- SED does not refer to a material plane, and depends only on the loading state.

Biaxiality and Crack Orientation

\[ \lambda = \theta \]

\[ \lambda_{12} = \lambda_1 = \lambda^B_1 \]

\[ \lim_{\lambda \to 1} (\log \lambda) = \varepsilon \]

\[ B = \frac{\varepsilon_2}{\varepsilon_1} \]

B = 1 equibiaxial tension
B = 0 pure shear
B = -0.5 simple tension
B = -1 pure in-plane shear

Plane Stress
Effect of Crack Orientation on CED

\[ W_c/W = \begin{cases} \lambda_1 & B = +1.00 \\ \lambda_2 = \lambda_1^B & \lambda_1 \leq B < 0 \\
\end{cases} \]

Crack Orientation Angle \( \theta \), degrees
Growth of Oriented Crack

![Diagram showing crack growth and orientation](image-url)
Exact Solution for Small Strains

\[ K_I(\theta) = K_I(\theta = 0^\circ) \left[ \cos^2 \theta + \beta \sin^2 \theta \right] \]

\[ K_{II}(\theta) = K_I(\theta = 0^\circ) \left[ 1 - \beta \right] \sin \theta \cos \theta \]

\[ \beta = \frac{\nu + B}{\nu B + 1} \] \text{ Stress Biaxiality}

\[ T = \left[ K_I^2 + K_{II}^2 \right] / E' \]

\( E' \) depends on whether conditions of plane stress or plane strain exist at the crack tip.

\[ E' = E \] \text{ plane stress}

\[ E' = \frac{E}{1 - \nu^2} \approx 0.75E \] \text{ plane strain} \quad \text{(LEFM Solution valid for } 0 < \beta < 1)
Exact Solution in terms of B and θ

\[
\frac{T(B, \theta)}{T(-0.5, 0^\circ)} = \frac{W_c(B, \theta)}{W} = \left. \frac{K_I^2(B, 0^\circ)}{K_I^2(-0.5, 0^\circ)} \right|_{W=\text{const}} \left( \cos^2 \theta + \beta^2 \sin^2 \theta \right)
\]

\[K_I = C \sigma_1 \sqrt{a}\]

\[
\left. \frac{K_I^2(B, 0^\circ)}{K_I^2(-0.5, 0^\circ)} \right|_{W=\text{const}} = \left. \left( \frac{\sigma_1(B)}{\sigma_1(B = -0.5)} \right)^2 \right|_{W=\text{const}}
\]

\[
\left. \left( \frac{\sigma_1(B)}{\sigma_1(B = -0.5)} \right)^2 \right|_{W=\text{const}} = \frac{1}{1 + \left( \frac{\nu + B}{\nu B + 1} \right)^2 - \frac{2\nu(\nu + B)}{\nu B + 1}}
\]
Comparison at Small Strains

LEFM*  
CED

Crack Plane Experiences Compression

CED
Scalar Parameters vs. CED

Shear Green-Lagrange Strain, $E_{12}$

Axial Green-Lagrange Strain, $E_{11}$
Scalar Parameters vs. CED

Shear Green-Lagrange Strain, $E_{12}$

Axial Green-Lagrange Strain, $E_{11}$
Scalar Parameters vs. CED
Scalar Parameters vs. CED

Axial Green-Lagrange Strain, $E_{11}$

Shear Green-Lagrange Strain, $E_{12}$
Continuum Parameters → Fatigue Life

\[ N_f = 16100 \left( W_{c,max} \right)^{-2.15} \]

\[ r^2 = 0.78 \]

\[ N_f = 25800 \left( W_{max} \right)^{-2.12} \]

\[ r^2 = 0.70 \]

\[ N_f = 10800 \left( \varepsilon_{1,max} \right)^{-3.52} \]

\[ r^2 = 0.83 \]
Cracking Plane Observations
Summary

• Multiaxial loading - how are loads applied to structure experienced by cracks at critical location?

• What is needed:
  – A plane-specific continuum parameter
    • SED, MP Stress, MP Strain are scalars
  – Ability to estimate energy release rate

• CED estimates portion of SED available for release in a given plane
  – Comparison with LEFM shows that CED can be improved

• Allows prediction of failure plane
## Equivalence Parameters for Nucleation Approach

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Physical Meaning</th>
<th>Robustness</th>
<th>Accessibility</th>
<th>WVM Recommendation</th>
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</thead>
<tbody>
<tr>
<td><strong>Scalar Parameters</strong></td>
<td></td>
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</tr>
<tr>
<td>Strain Energy Density</td>
<td>• Not all stored energy is available for release on a given plane</td>
<td>• Worst correlation across range of deformation states</td>
<td>• Easily calculated from FEA</td>
<td>• Do Not Use</td>
</tr>
<tr>
<td>Max Principal Strain</td>
<td>• For many load histories, cracks occur on plane experiencing greatest tensile strain</td>
<td>• Good correlation across range of deformation states</td>
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</tr>
<tr>
<td>Max Principal Stress</td>
<td>• For many load histories, cracks occur on plane experiencing greatest tensile stress</td>
<td>• Santier and others have successfully applied this parameter over a range of deformation states</td>
<td>• Easily calculated from FEA</td>
<td>• Useful for simple load histories</td>
</tr>
<tr>
<td><strong>Plane-Specific Parameters</strong></td>
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</tr>
<tr>
<td>Cracking Energy Density</td>
<td>• Energy density available on specific material plane</td>
<td>• Good correlation across range of deformation states</td>
<td>• Requires expensive calculation to perform numerical integration / search for critical plane</td>
<td>• Can be improved on, but probably provides truest picture yet when loading is very complex</td>
</tr>
</tbody>
</table>

- **Max Principal Strain**
- **Max Principal Stress**
- **Cracking Energy Density**